

## 3.2 Semantics of the Lambda Calculus

Mittwoch, 8. Juni 2016 11:00

We now define the operational semantics of the lambda calculus by introducing an interpreter for lambda terms that uses 3 reduction rules:  $\alpha$ -reduction,  $\beta$ -reduction,  $\delta$ -reduction

$\alpha$ -reduction: renaming bound variables

Def 3.2.1 ( $\alpha$ -Reduction) (Slide 54)

The relation  $\rightarrow_\alpha \subseteq \Lambda \times \Lambda$  is the smallest relation with

- $\lambda x. t \rightarrow_\alpha \lambda y. t[x/y]$  if  $y \notin \text{free}(t)$
- if  $t_1 \rightarrow_\alpha t_2$ , then  $(t_1 r) \rightarrow_\alpha (t_2 r)$ ,  
 $(r t_1) \rightarrow_\alpha (r t_2)$ , and  $(\lambda y. t_1) \rightarrow_\alpha (\lambda y. t_2)$ .

Ex:

$$\lambda x y. x y \rightarrow_\alpha \lambda x z. x z$$

$$\rightarrow_\alpha \lambda v z. v z$$

$$\rightarrow_\alpha \dots$$

$$\lambda x. x y \not\rightarrow_\alpha \lambda x. x z$$

No renaming  
of free  
variables

$\beta$ -reduction is used to apply a  $\lambda$ -abstraction to a term:  $(\lambda x. t) r$  results in  $t$  where all free occurrences of  $x$  are replaced by  $r$

Def 3.2.2 ( $\beta$ -Reduction) (Slide 54)

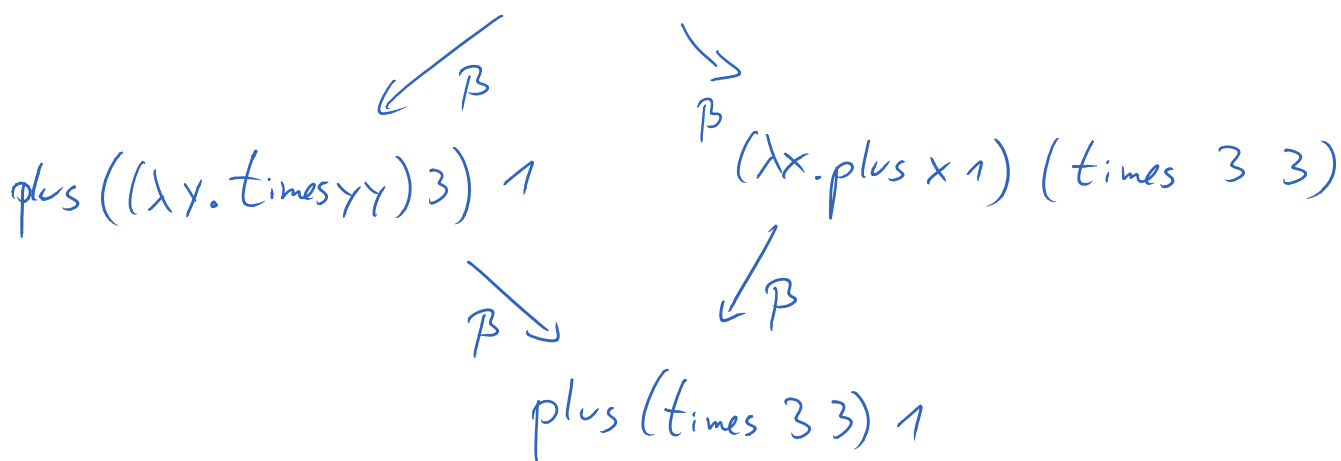
The relation  $\rightarrow_{\beta} \subseteq \Lambda \times \Lambda$  is the smallest relation with

- $(\lambda x. t) r \rightarrow_{\beta} t [x/r]$
- if  $t_1 \rightarrow_{\beta} t_2$ , then  $(t_1 r) \rightarrow_{\beta} (t_2 r)$ ,  $(r t_1) \rightarrow_{\beta} (r t_2)$ , and  $(\lambda y. t_1) \rightarrow_{\beta} (\lambda y. t_2)$

Ex:  $(\lambda x. x) \text{zero} \rightarrow_{\beta} x [x/\text{zero}] = \text{zero}$

$(\lambda x y. x y) y \rightarrow_{\beta} (\lambda y. x y) [x/y] = \lambda y'. y y'$

$(\lambda x. \text{plus } x \ 1) ((\lambda y. \text{times } y \ y) \ 3)$



There can be several possibilities to evaluate a

$\lambda$ -term by  $\beta$ -reduction. Do they always yield the same result?

Def 3.2.3 (transitive-reflexive closure, normal form, confluence)

Let  $\rightarrow$  be a relation on some set  $N$ .

(a) The transitive-reflexive closure  $\rightarrow^*$  is the smallest relation such that:

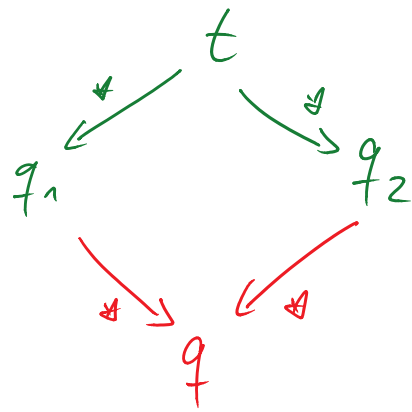
- $t_1 \rightarrow t_2$  implies  $t_1 \rightarrow^* t_2$
- $t_1 \rightarrow t_2 \rightarrow^* t_3$  implies  $t_1 \rightarrow^* t_3$
- $t_1 \rightarrow^* t_1$

In other words:  $t_0 \rightarrow^* t_n$  iff  $t_0 \rightarrow t_1 \rightarrow t_2 \rightarrow \dots \rightarrow t_n$   
for  $n \geq 0$

(b) An object  $q \in N$  is a normal form iff there is no  $q' \in N$  with  $q \rightarrow q'$ .

We say that  $q$  is a normal form of  $t$  iff  $t \rightarrow^* q$  and  $q$  is a normal form.

(c) The relation  $\rightarrow$  is confluent iff for all  $t, q_1, q_2 \in N$  we have: if  $t \rightarrow^* q_1$  and  $t \rightarrow^* q_2$ , then there exists a  $q \in N$  with  $q_1 \rightarrow^* q$  and  $q_2 \rightarrow^* q$ .

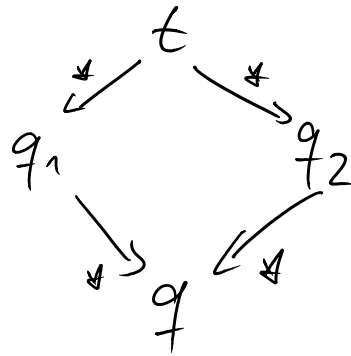


if the green part holds,  
then the red part holds as well.

Lemma 3.2.4 (Confluence implies unique normal forms)

Let  $\rightarrow$  be a confluent relation on a set  $N$ . Then every  $t \in N$  has at most one normal form.

Proof: Let  $t \in N$  have 2 normal forms  $t_1, t_2$ .



By confluence, there exists a  $q \in N$  with  $t_1 \rightarrow q, t_2 \rightarrow q$ .

Since  $t_1, t_2$  are normal forms, we have

$$t_1 = q = t_2.$$

□

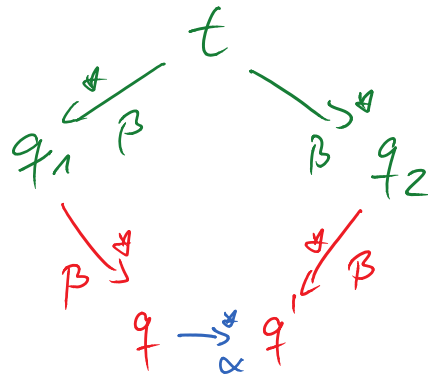
Thm 3.2.5 (Confluence of the  $\lambda$ -calculus with  $\beta$ -reduction, Church & Rosser)

$\rightarrow_{\beta}$  is confluent, i.e.,

if  $t \rightarrow_{\beta}^* t_1$  and  $t \rightarrow_{\beta}^* t_2$ ,

then there exist  $q, q' \in \Lambda$  with

$$t_1 \rightarrow_{\beta}^* q, t_2 \rightarrow_{\beta}^* q', \text{ and } q \rightarrow_{\beta}^* q'$$



The last form of reduction rules in the  $\lambda$ -calculus is needed to evaluate terms built with constants from  $\mathcal{C}$ . In particular, these constants could correspond to pre-defined functions of Haskell.

But: we want to make sure that  $\delta$ -reduction does not destroy the confluence of the  $\lambda$ -calculus.

Solution: Define  $\delta$ -reduction by a set of  $\delta$ -rules. These rules must have a certain restricted form which ensures that evaluation with  $\beta$ - and  $\delta$ -reduction remains confluent.

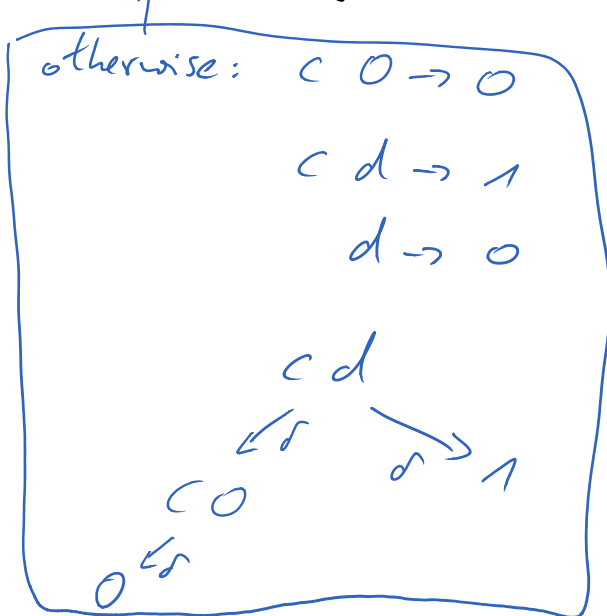
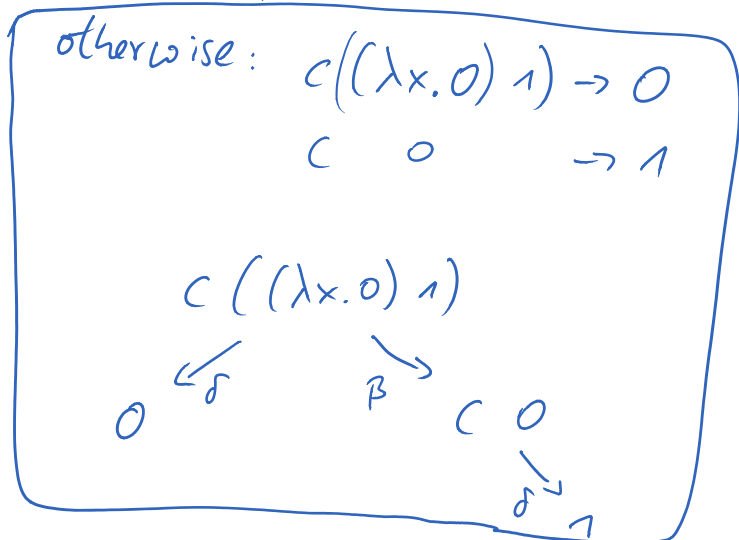
Def 326 ( $\delta$ -Reduction) (Slide 55)

A set of rules  $\delta$  of the form  $c t_1 \dots t_n \rightarrow r$  with  $c \in \mathcal{C}$ ,  $t_1, \dots, t_n, r \in \Lambda$  is a Delta-Rule-Set iff  $t_1, \dots, t_n, r$  are closed  $\lambda$ -terms

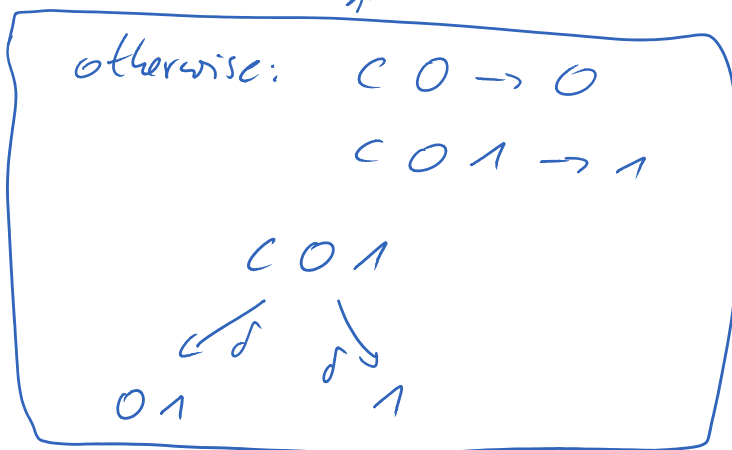
otherwise:  $\left. \begin{array}{l} c 0 \rightarrow 0 \\ c x \rightarrow 1 \end{array} \right\} \delta$   
 Then  $\begin{array}{c} c 0 \\ \swarrow \delta \quad \searrow \end{array}$



•  $t_1, \dots, t_n$  are in  $\rightarrow_{\beta}$ -normal form and they do not contain any left-hand side of a rule from  $\delta$



• In  $\delta$ , there are no two rules  $ct_1 \dots t_n \rightarrow r$ ,  $ct_1 \dots t_n \dots t_m \rightarrow r'$  with  $m \geq n$ .



For such a set  $\delta$ ,  $\rightarrow_{\delta}$  is the smallest relation with

- $l \rightarrow_{\delta} r$  for all  $l \rightarrow r \in \delta$
- if  $t_1 \rightarrow_{\delta} t_2$ , then  $(t_1 r) \rightarrow_{\delta} (t_2 r)$ ,  $(r t_1) \rightarrow_{\delta} (r t_2)$ , and  $\lambda y. t_1 \rightarrow_{\delta} \lambda y. t_2$

We define  $\rightarrow_{\beta\delta} = \rightarrow_{\beta} \cup \rightarrow_{\delta}$ .

Ex. for a Delta-Rule-Set:

$\delta = \{ \text{isa}_{\text{Succ}} (\text{Succ } t) \rightarrow \text{True} \mid t \in \underline{A}, t \text{ is closed, and in } \rightarrow_{\beta\delta}\text{-normal form} \} \cup$

$\{ \text{isa}_{\text{Succ}} \text{ zero} \rightarrow \text{False} \}$

Thm 3.2.7 (Confluence for the  $\lambda$ -calculus with  $\rightarrow_{\beta}$  and  $\rightarrow_{\delta}$ )

$\rightarrow_{\beta\delta}$  is confluent, i.e.,

if  $t \xrightarrow{\delta}_{\beta\delta} q_1$  and  $t \xrightarrow{\delta}_{\beta\delta} q_2$ ,

then there exist  $q, q' \in \underline{A}$  with

$q_1 \xrightarrow{\delta}_{\beta\delta} q$ ,  $q_2 \xrightarrow{\delta}_{\beta\delta} q'$ , and  $q \xrightarrow{\delta}_{\alpha} q'$ .

---

So  $\rightarrow_{\beta\delta}$  defines an operational semantics for the  $\lambda$ -calculus (and  $\rightarrow_{\alpha}$  defines that we regard certain terms as being "equal").

( A denotational semantics for the  $\lambda$ -calculus  
was only invented 30 years later by D. Scott.  
Crucial idea: continuous functions ).